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2002 J. Phys. A: Math. Gen. 35 3279

(http://iopscience.iop.org/0305-4470/35/14/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3279–3287 PII: S0305-4470(02)31657-3

Darboux transformations of the one-dimensional stationary Dirac equation

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Received 3 December 2001, in final form 25 February 2002
Published 29 March 2002
Online at stacks.iop.org/JPhysA/35/3279

Abstract

A matricial Darboux operator intertwining two one-dimensional stationary Dirac Hamiltonians is constructed. This operator is such that the potential of the second Dirac Hamiltonian, as well as the corresponding eigenfunctions, are determined through the knowledge of only two eigenfunctions of the first Dirac Hamiltonian. Moreover this operator, together with its adjoint and the two Hamiltonians, generate a quadratic deformation of the superalgebra subtending the usual supersymmetric quantum mechanics. Our developments are illustrated in the free-particle case and the generalized Coulomb interaction. In the latter case, a relativistic counterpart of shape invariance is observed.

PACS numbers: 02.30.Jr, 02.30.Tb, 03.65.−w, 11.30.Pb

1. Introduction

In quantum mechanics, the Schrödinger equations which can be solved by analytic methods exclusively are rather exceptional. Therefore those methods able to enlarge the number of such equations have attracted much attention in recent as well as less recent literature. Three of them still remain very popular: the Darboux transformations [1] elaborated in 1882 within the mathematical framework of Sturm–Liouville differential equations, the factorization method introduced by Schrödinger [2] in 1940 and more recently the so-called supersymmetric quantum mechanics [3]. All of them are more or less based on the following ideas.

Let us consider the following Schrödinger Hamiltonian

$$H_0 \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \qquad x \in \mathbb{R} \text{ or } x \in \mathbb{R}_0^+ \tag{1}$$

which can be factorized as follows:

$$H_0 = L^{\dagger} L + \alpha, \qquad \alpha = \text{constant} \tag{2}$$

with

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{3}$$

Then the eigenfunctions of the isospectral (up to the eventual creation or loss of one energy) Hamiltonian H_1 defined by

$$H_1 \equiv LL^{\dagger} + \alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x) \tag{4}$$

are obtained through the application of L to the eigenfunctions of H_0 as it is clear from the so-called intertwining relation

$$LH_0 = H_1 L. \tag{5}$$

Thus a new Schrödinger Hamiltonian H_1 has been constructed and it is exactly solvable if H_0 is.

We remark that the Darboux transformation has two particular features compared to [2,3]. First, the potential $W(x)$, written as

$$W(x) = -\frac{\mathrm{d}\,\ln\psi_0(x)}{\mathrm{d}x}, \tag{6}$$

can be constructed from a bounded *or unbounded* non-vanishing eigenfunction $\psi_0(x)$ of H_0 (with eigenvalue α). Second, the Darboux operator L can be extended to higher order [1,4].

Here we shall ask for the same kind of developments in the relativistic context, that is to say, search for the operator L intertwining two one-dimensional Dirac Hamiltonians. A partial answer has already been given in [5–8] through the supersymmetrical features of specific Dirac Hamiltonians. Another one can also be found in [9, 10] where a relativistic Darboux transformation has been considered but for pseudoscalar potentials only. In the following we will not limit ourselves to such a context and will give, in section 2, the extended intertwining operator L corresponding to a general self-adjoint potential. This operator is constructed from two (known) solutions of the initial Dirac equation and gives rise to new exactly solvable Dirac equations. Moreover, in section 3, we will convince ourselves from this operator L and its adjoint that the underlying superstructure in the relativistic context is a quadratic *deformation* of the $sqm(2)$ superalgebra, the latter being, as is well known [3], the subtending superalgebra of the (non-relativistic) supersymmetric quantum mechanics. Finally, in section 4, we will illustrate our statements by two examples: the free-particle case and the generalized Coulomb interaction. For the latter, we observe the relativistic counterpart of the so-called shape invariance [11], i.e. only the values of the parameters introduced in the expression of the potentials change.

2. Intertwining operator for the Dirac equation

Let us start with the following one-dimensional Dirac Hamiltonian:

$$h_0 \equiv \mathrm{i}\sigma_2 \frac{\mathrm{d}}{\mathrm{d}x} + v_0(x), \qquad x \in \mathbb{R} \text{ or } x \in \mathbb{R}_0^+ \tag{7}$$

where σ_2 is the usual 2×2 Pauli matrix and v_0 is real and symmetric, i.e.

$$v_0(x) = \begin{pmatrix} v_{11}^0(x) & v_{12}^0(x) \\ v_{12}^0(x) & v_{22}^0(x) \end{pmatrix}. \tag{8}$$

We assume here that h_0 is a known exactly solvable Hamiltonian; in other words, all its eigenfunctions, the two-component spinors $\psi(x)$, as well as the corresponding energies are analytically determined. Let us now search for a matricial operator L satisfying the intertwining relation similar to (5), i.e.

$$Lh_0 = h_1 L \tag{9}$$

with

$$h_1 \equiv \mathrm{i}\sigma_2 \frac{\mathrm{d}}{\mathrm{d}x} + v_1(x), \tag{10}$$

$v_1(x)$ at this level being the unknown real and symmetric potential. The simplest operator L we can consider is

$$L \equiv A\frac{\mathrm{d}}{\mathrm{d}x} + B \tag{11}$$

where A and B are 2×2 matrices with x-dependent entries. The relations (9) and (11) give the following system:

$$[A, \sigma_2] = 0, \tag{12}$$

$$[B, \sigma_2] - \mathrm{i}Av_0 + \mathrm{i}v_1 A - \sigma_2 A_x = 0, \tag{13}$$

$$Av_{0x} + Bv_0 - v_1 B - \mathrm{i}\sigma_2 B_x = 0, \tag{14}$$

the notation A_x meaning here

$$\frac{\mathrm{d}A}{\mathrm{d}x} \equiv \begin{pmatrix} \frac{\mathrm{d}A_{11}}{\mathrm{d}x} & \frac{\mathrm{d}A_{12}}{\mathrm{d}x} \\ \frac{\mathrm{d}A_{21}}{\mathrm{d}x} & \frac{\mathrm{d}A_{22}}{\mathrm{d}x} \end{pmatrix}.$$

The condition (12) is equivalent to asking for $A_{11} = A_{22}$ and $A_{12} = -A_{21}$. The constraint (13) enables us to fix the potential difference $\Delta v \equiv v_1 - v_0$:

$$\Delta v = (Av_0 - v_0 A + \mathrm{i}[B, \sigma_2] - \mathrm{i}\sigma_2 A_x)A^{-1} \tag{15}$$

up to the assumption of the existence of A^{-1}. Finally, from (14) we can obtain the matrix B or, in a simpler way σ, defined through $B \equiv A\sigma$. Indeed, equation (14) is then

$$(v_0 - \mathrm{i}\sigma_2\sigma)_x + [\sigma, v_0] + \mathrm{i}[\sigma_2, \sigma]\sigma = 0. \tag{16}$$

We recognize a matrix analogue of the Riccati equation. It can be linearized through the substitution

$$\sigma = -u_x u^{-1} \tag{17}$$

in order to become

$$[u^{-1}(v_0 u + \mathrm{i}\sigma_2 u_x)]_x = 0 \tag{18}$$

which after integration leads to

$$h_0 u = \mathrm{i}\sigma_2 u_x + v_0 u = u\lambda, \tag{19}$$

the matrix λ being the constant of integration.

This equation (19) is thus, formally speaking, an ordinary Dirac one up to the fact that the solution u is no longer a spinor but a 2×2 matrix while the usual energy E has also been replaced by a 2×2 matrix λ.

The next step is to find a convenient u that is a solution of (19) being real (and invertible) in order to ensure the self-adjointness of v_1 through (15). It is ensured in a straightforward manner if

$$u = (u_1, u_2), \qquad \lambda = \mathrm{diag}(\varepsilon_1, \varepsilon_2) \tag{20}$$

with the spinors u_1 and u_2 being eigenfunctions (not necessarily bounded) of the Dirac Hamiltonian h_0:

$$h_0 u_j = \varepsilon_j u_j, \qquad j = 1, 2. \tag{21}$$

Having found u, the operator L given in equation (11) or

$$L = A\left(\frac{\mathrm{d}}{\mathrm{d}x} - u_x u^{-1}\right) \tag{22}$$

as well as the new potential v_1 (see equation (15)):

$$v_1 = A(v_0 + \mathrm{i}[\sigma, \sigma_2] - \mathrm{i}\sigma_2 A^{-1} A_x)A^{-1} \tag{23}$$

are now fixed up to the determination of A. This matrix keeps arbitrariness: all one knows is that it has to commute with σ_2. For simplicity and comparison with the non-relativistic context, we put A equal to the identity matrix. Equations (22) and (23) are then simplified as follows:

$$L = \frac{\mathrm{d}}{\mathrm{d}x} - u_x u^{-1}, \tag{24}$$

$$v_1 = v_0 + \mathrm{i}[\sigma, \sigma_2]. \tag{25}$$

These results are the relativistic analogues of the usual Darboux transformation. We now give another expression for what concerns v_1, particularly useful for applications. Indeed, from (19) we have

$$\sigma = -u_x u^{-1} = \mathrm{i}\sigma_2 u\lambda u^{-1} - \mathrm{i}\sigma_2 v_0 \tag{26}$$

and therefore

$$v_1 = \sigma_2 v_0 \sigma_2 + u\lambda u^{-1} - \sigma_2 u\lambda u^{-1}\sigma_2, \tag{27}$$

i.e.

$$v_1 = \sigma_2 v_0 \sigma_2 + \frac{\varepsilon_1 - \varepsilon_2}{\det u}\begin{pmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{pmatrix}, \tag{28}$$

where $d_1 \equiv u_{11}u_{22} + u_{12}u_{21}$, $d_2 = u_{21}u_{22} - u_{11}u_{12}$, with u_{ij} corresponding to the element of the matrix u at the crossing of the ith line and the jth column.

Let us close this section by noticing that, by definition, the operator L has a non-trivial kernel since $\ker L = u$. This implies that the action of L to an eigenspinor of h_0 corresponding to an eigenvalue different from ε_1 and ε_2 will give rise to an eigenspinor of h_1. The eigenspinors of h_1 related to the eigenvalues ε_1 and ε_2 will be obtained through $v \equiv (u^\dagger)^{-1}$, that is $h_1 v = v\lambda$.

3. Dirac Hamiltonians and second-order supersymmetry

Let us consider here, in addition to L given in equation (24), its adjoint $L^\dagger$ defined by

$$L^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} - (u_x u^{-1})^\dagger. \tag{29}$$

It satisfies an intertwining relation similar to equation (9):

$$L^\dagger h_1 = h_0 L^\dagger. \tag{30}$$

This relation means that the operator $L^\dagger$ realizes the transformation in the opposite direction, i.e. the application of $L^\dagger$ to the eigenspinors of h_1 gives us the eigenspinors of h_0. The operator $L^\dagger L$ is thus such that, applied to the eigenspinors of h_0, it gives back these eigenspinors. By definition, this is nothing other than the fact that $L^\dagger L$ is a symmetry operator of the initial Dirac equation $h_0\psi = E\psi$. Moreover, because $L^\dagger L$ is a second-order differential (matricial) operator while h_0 is of the first order, $L^\dagger L$ is, in fact, a polynomial of second order in h_0. More precisely, after tedious calculations, one can be convinced that

$$L^\dagger L = (h_0 - \varepsilon_1)(h_0 - \varepsilon_2) \tag{31}$$

while a similar result holds for $LL^\dagger$:

$$LL^\dagger = (h_1 - \varepsilon_1)(h_1 - \varepsilon_2). \tag{32}$$

If we now introduce the 4×4 matrices

$$H \equiv \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix} \tag{33}$$

the relations (9) and (30)–(32) can be reformulated as

$$[Q, H] = [Q^\dagger, H] = 0, \qquad \{Q, Q^\dagger\} \equiv QQ^\dagger + Q^\dagger Q = (H - \varepsilon_1)(H - \varepsilon_2) \tag{34}$$

while

$$Q^2 = (Q^\dagger)^2 = 0. \tag{35}$$

Relations (34) and (35) are those of a quadratic deformation of the superalgebra sqm(2) subtending the usual supersymmetric quantum mechanics [3]. This quadratic superalgebra cannot be seen directly from the Dirac equation and therefore we associate it with a hidden supersymmetry. Let us also finally notice that a superalgebra similar to the one of (34) and (35) can also be found in the non-relativistic context when considering second-order Darboux transformations [12].

4. Examples

Let us now turn to some examples and see how our method provides us with new exactly solvable Dirac potentials from known ones.

4.1. The free-particle case

We consider here the potential

$$v_0(x) = m\sigma_1, \qquad x \in \mathbb{R}. \tag{36}$$

Note that it corresponds to an unusual—but convenient—realization (the usual one being associated with $v_0(x) = m\sigma_3$) of the Clifford algebra subtending the one-dimensional Dirac equation. As stated in (20), it is necessary to take account of two eigenspinors corresponding to (36). Let u_1 and u_2, defined by

$$u_1 = \begin{pmatrix} \mathrm{ch}(kx) + \frac{cE}{k}\mathrm{sh}(kx) \\ \mathrm{ch}(kx + 2\alpha) + \frac{cE}{k}\mathrm{sh}(kx + 2\alpha) \end{pmatrix}, \qquad u_2 = \begin{pmatrix} -\mathrm{ch}(kx) \\ \mathrm{ch}(kx + 2\alpha) \end{pmatrix}, \tag{37}$$

be such eigenfunctions (of respective eigenvalues $\varepsilon_1 = E$ and $\varepsilon_2 = -E$) with

$$k = \sqrt{m^2 - E^2}, \qquad \mathrm{e}^{2\alpha} = \sqrt{\frac{m-k}{m+k}}, \qquad c = \text{constant}. \tag{38}$$

The unique constraint to take care of in order to apply our method is to have a non-vanishing determinant: $\det u \neq 0$. Here it is precisely given by

$$\det u = \frac{1}{E}\left[m + E\,\mathrm{ch}(2kx + 2\alpha) + \frac{E^2 c}{k}\mathrm{sh}(2kx + 2\alpha)\right] \equiv \frac{1}{E}\Delta \tag{39}$$

and the parameter c is such that $|c| < k/E$ in order to satisfy this constraint. The result (28) then gives rise to the new exactly solvable potential v_1:

$$v_1(x) = \frac{2E^2c}{\Delta}\sigma_3 + \left(m - \frac{2k^2}{\Delta}\right)\sigma_1 \tag{40}$$

whose eigenspinors can be obtained from the application of L defined in equation (24) to the eigenspinors of the free Dirac Hamiltonian. Notice that the potential $v_1(x)$ given in equation (40) reduces to the well known one-soliton scalar potential when $c = 0$.

4.2. The generalized Coulomb case

Before proceeding to this example, we would like to mention that the usual radial equation associated with the (3+1)-dimensional Dirac equation is included in our developments. Indeed, the standard radial equation, when coupled to scalar $W(x)$ and vector $V(x)$ potentials, is

$$\left\{\frac{\mathrm{d}}{\mathrm{d}x} - \frac{k}{x}\sigma_3 + [M + W(x)]\sigma_1 + \mathrm{i}[E - V(x)]\sigma_2\right\}\psi(x) = 0, \qquad x \in \mathbb{R}_0^+ \tag{41}$$

where M and E are the mass and the energy of the particle while k is related to the total angular momentum. Equation (41) can also be written as

$$\left\{\mathrm{i}\sigma_2\frac{\mathrm{d}}{\mathrm{d}x} + \frac{k}{x}\sigma_1 + [M + W(x)]\sigma_3 - [E - V(x)]\right\}\psi(x) = 0, \tag{42}$$

which coincides with $h_0\psi(x) = E\psi(x)$ with h_0 defined in equation (7) and

$$v_{12}^0(x) = \frac{k}{x}, \qquad v_{11}^0(x) = M + V(x) + W(x), \qquad v_{22}^0(x) = -M + V(x) - W(x). \tag{43}$$

Let us now turn to our example. It corresponds to the choices of [13]:

$$V(x) = \frac{\alpha}{x}, \qquad W(x) = \frac{\beta}{x}. \tag{44}$$

We refer to this example as the generalized Coulomb one because the choice ($\alpha = \frac{1}{137}$, $\beta = 0$) leads to the standard Coulomb interaction.

Let $\psi(x) = (\psi_1(x), \psi_2(x))^T$ be a solution of equation (41) or equivalently (42), when the interactions (44) are taken into account. Using standard developments, we easily find the solutions in terms of hypergeometric confluent functions

$$\begin{aligned}\psi_1(x) = \mathrm{e}^{-\lambda_n x}x^{\mu}\Bigg[&-n\ {}_1F_1(1-n, 2\mu+1; 2\lambda_n x)\\ &-\left(-k + \frac{\alpha M}{\lambda_n} + \frac{\beta E_n}{\lambda_n}\right){}_1F_1(-n, 2\mu+1; 2\lambda_n x)\Bigg],\end{aligned} \tag{45}$$

$$\begin{aligned}\psi_2(x) = -\frac{\lambda_n}{M + E_n}\mathrm{e}^{-\lambda_n x}x^{\mu}\Bigg[&-n\ {}_1F_1(1-n, 2\mu+1; 2\lambda_n x)\\ &+\left(-k + \frac{\alpha M}{\lambda_n} + \frac{\beta E_n}{\lambda_n}\right){}_1F_1(-n, 2\mu+1; 2\lambda_n x)\Bigg],\end{aligned} \tag{46}$$

where the parameters λ_n and μ are constrained by

$$\lambda_n^2 = M^2 - E_n^2, \tag{47}$$

$$\mu^2 = k^2 + \beta^2 - \alpha^2 \tag{48}$$

while the number n is defined by

$$n = -\left(\frac{\alpha E_n}{\lambda_n} + \frac{\beta M}{\lambda_n} + \mu\right). \tag{49}$$

This relation can be solved for the energies E_n as

$$E_n = \frac{-\alpha\beta \pm (n+\mu)\sqrt{\alpha^2 + (n+\mu)^2 - \beta^2}}{[\alpha^2 + (n+\mu)^2]} M, \tag{50}$$

the plus or minus sign, as well as the values taken by n, having possibly to be chosen in order to ensure the square-integrability of $\psi_1(x)$ and $\psi_2(x)$.

The most straightforward way to apply our method is to choose

$$u_1 = \left.\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\right|_{n=0}, \qquad u_2 = \left.\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\right|_{n=1}. \tag{51}$$

In order to avoid heavy notation, we rewrite these choices as

$$u_1 = \begin{pmatrix} x^{\mu} \mathrm{e}^{-\lambda_0 x} \\ c_1 x^{\mu} \mathrm{e}^{-\lambda_0 x} \end{pmatrix}, \qquad u_2 = \begin{pmatrix} x^{\mu} \mathrm{e}^{-\lambda_1 x}(1 - c_2 x) \\ c_1 x^{\mu} \mathrm{e}^{-\lambda_1 x}(1 - c_3 x) \end{pmatrix} \tag{52}$$

with λ_0 and λ_1 defined through equations (49) and (50), while

$$c_1 = \frac{\mu - k}{\alpha - \beta}, \tag{53}$$

$$c_2 = \frac{\lambda_1}{1+2\mu} + \frac{(E_1 + M)(\mu - k)}{(\alpha - \beta)(1+2\mu)}, \tag{54}$$

$$c_3 = \frac{\lambda_1}{1+2\mu} + \frac{(M - E_1)(\alpha - \beta)}{(\mu - k)(1+2\mu)}. \tag{55}$$

Applying finally the result (28), we obtain a (new) exactly solvable potential of the type

$$v_1(x) = \frac{\alpha}{x} + \left[-M + \frac{(\varepsilon_1 - \varepsilon_2)}{c_2 - c_3}\left(\frac{2}{x} - c_2 - c_3\right)\right]\sigma_3 \\ + \left\{-\frac{k}{x} + \frac{(\varepsilon_1 - \varepsilon_2)}{c_2 - c_3}\left[\left(c_1 - \frac{1}{c_1}\right)\frac{1}{x} + \left(\frac{c_2}{c_1} - c_1 c_3\right)\right]\right\}\sigma_1. \tag{56}$$

In other words, we obtain a shape-invariant potential with respect to $v_0(x)$.

A particular example corresponding to the choices

$$\alpha = 1, \qquad \beta = -1, \qquad \mu = 1, \qquad k = 1 \tag{57}$$

can be useful to illustrate the results here. Indeed we have

$$\lambda_0 = 0, \qquad \lambda_1 = \tfrac{4}{5}M, \qquad c_1 = 0, \qquad c_2 = \tfrac{4}{15}M, \qquad c_1 c_3 = \tfrac{8}{15}M. \tag{58}$$

The corresponding energies are

$$\varepsilon_1 \equiv E_0 = M, \qquad \varepsilon_2 \equiv E_1 = -\tfrac{3}{5}M, \tag{59}$$

ensuring that the term $(\alpha M + \beta E_n)/\lambda_n$ in equations (45) and (46) with $n = 0$ simply vanishes. Indeed, with the parameters α and β of equation (57), this term amounts to $\sqrt{(M - E_n)/(M + E_n)}$. For $n = 0$, this is vanishing since $E_0 = M$ from equation (59). The resulting potential is given by equation (56), i.e.

$$v_1(x) = \frac{1}{x} + \left(\frac{3M}{5} + \frac{1}{x}\right)\sigma_3 + \left(\frac{2}{x} - \frac{4M}{5}\right)\sigma_1. \tag{60}$$

One can then determine the operator L, as defined in equation (24), connecting the eigenfunctions related to $v_0(x) = (1/x) + (M - 1/x)\sigma_3 + (1/x)\sigma_1$ and to $v_1(x)$ given in equation (60), respectively. It is given by

$$L = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x} & \frac{2M}{5} - \frac{2}{x} \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} + \frac{4M}{5} - \frac{2}{x} \end{pmatrix}. \tag{61}$$

Due to its definition, it is clear that $Lu_1 \equiv Lu_2 = 0$, with u_1 and u_2 of equation (52) with the values (58). The definition of L also implies that, whenever applied to any of the functions $\psi(x)$, it will give the eigenfunctions corresponding to $v_1(x)$ as expressed in equation (60). For instance, for $n = 2$, we have

$$L\left[\left(-\frac{2}{25}\exp^{-(3M/5)x}\right)\begin{pmatrix} 50x - 30Mx^2 + 3M^2x^3 \\ 3(-10Mx^2 + 3M^2x^3) \end{pmatrix}\right] = -\frac{6}{125}\exp^{-(3M/5)x} M^2x^2 \begin{pmatrix} -10 + 3Mx \\ 5 + 3Mx \end{pmatrix}, \tag{62}$$

and one can directly check that this is a solution of the final equation $h_1\psi(x) = E\psi(x)$ with $E = -\frac{4}{5}M$. The other values of n $(= 3, 4, \ldots)$ evidently lead to similar results.

The last information we mention here is the possibility of obtaining new exactly solvable potentials and not only shape-invariant ones. This situation arises for example when we choose

$$u_1 = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\Bigg|_{n=1}, \qquad u_2 = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\Bigg|_{n=2}. \tag{63}$$

With the set of parameters (57), we obtain

$$v_1(x) = \frac{1}{50x - 15Mx^2 + 12M^2x^3} \times \begin{pmatrix} 100 + 90Mx - 60M^2x^2 & 100 - 115Mx - 27M^2x^2 + 12M^3x^3 \\ 100 - 115Mx - 27M^2x^2 + 12M^3x^3 & -120Mx + 84M^2x^2 \end{pmatrix} \tag{64}$$

which has a different shape with respect to $v_0(x)$ and to the $v_1(x)$ given in equation (60). Evidently, one can determine the eigenfunctions related to this potential $v_1(x)$ defined in equation (64) through the application of the corresponding Darboux operator L on the eigenfunctions $\psi(x)$ of h_0. One can also proceed in a similar way with different values of n and obtain families of new exactly solvable potentials $v_1(x)$, whose eigenfunctions will be known through the application of the *ad hoc* Darboux operator on the solutions of the generalized Coulomb problem.

Acknowledgments

The work of BFS was partially supported by a grant of the Russian Foundation for Basic Research and a 'Bourse de Séjour Scientifique' from the FNRS, Belgium. The work of ND and BVdB was supported by the Institut Interuniversitaire des Sciences Nucléaires de Belgique.

References

[1] Darboux G 1882 *C. R. Acad. Sci., Paris* **94** 1343
Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[2] Schrödinger E 1940 *Proc. Irish Acad.* A **46** 9
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[3] Witten E 1981 *Nucl. Phys.* B **185** 513
[4] Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys.* A **10** 2683
Bagrov V G and Samsonov B F 1995 *Theor. Math. Phys.* **104** 1051
[5] Toyama F M and Nogami Y 1998 *Phys. Rev.* A **57** 93
[6] Hugues R J, Kostelecky V A and Nieto M M 1986 *Phys. Rev.* D **34** 1100
[7] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** L697
[8] Nogami Y and Toyama F M 1993 *Phys. Rev.* A **47** 1708
[9] Stahlhofen A A 1994 *J. Phys. A: Math. Gen.* **27** 8279
[10] Anderson A 1991 *Phys. Rev.* A **43** 4602

[11] Gendenshtein L E 1983 *JETP Lett.* **38** 356

[12] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 951
Beckers J, Debergh N and Gotti C 1998 *Helv. Phys. Acta* **71** 214
Andrianov A, Ioffe M and Spiridonov V 1993 *Phys. Lett.* A **174** 273
Klishevich S and Plyushchay M 1999 *Mod. Phys. Lett.* A **14** 2739
Plyushchay M 2000 *Int. J. Mod. Phys.* A **15** 3679
Aoyama H, Sato M and Tanaka T 2001 *Phys. Lett.* B **503** 423
Aoyama H, Sato M and Tanaka T 2001 *Nucl. Phys.* B **619** 105

[13] Brihaye Y and Kosinski P 1998 *Mod. Phys. Lett.* A **13** 1445